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ANALYTICAL SOLUTION TO THE TWO DIMENSIONAL TRANSIENT BIOHEAT EQUATION WITH CONVECTIVE BOUNDARY CONDITIONS

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Abstract. *In this paper, the classical integral transform technique is applied to solve the two dimensional Pennes bioheat transfer equation in Cartesian coordinates subjected to convective boundary conditions. A straightforward analytical solution is obtained allowing for benchmark results and furnishing a close insight of some relevant aspects of cancer treatment by hyperthermia.*

Keywords. *hyperthermia, Pennes equation, integral transform technique.*

1. Introduction

The success of the treatment of malignant tumors by hyperthermia calls for the knowledge of the temperature field for both healthy and cancerous tissues. This medical procedure consists in exposing malignant cells to temperatures in the range of 42 °C to 45 °C in order to retard, arrest or reverse the growth of tumors. Although susceptible to the increase in temperature, the normal cells do not exhibit the same degree of sensitivity as the malignant cells and therefore the potential of induced hyperthermia as a therapy for cancer has been long considered for both local and global treatments. For the case of local hyperthermia, it is important to predict and control the thermal fields generated by the external medical device responsible for increasing local temperature. However, the task of controlling both temperature levels and duration of the heating process cannot be satisfactory accomplished based solely on measurements because only a few localized temperature readings can be monitored during the therapy. Therefore, mathematical models commonly employed in engineering are often used in the simulation of heat transfer in living tissues, furnishing temperature profiles that may guide the physician before and after the treatment. One of these models is the bioheat transfer equation proposed by Pennes (1948) that consists in a heat diffusion equation together with an energy sink term that accounts for the effect of the temperature difference between the blood supply and the tissue. The source term of this equation carries the effects of both metabolic heat generation and the external heat source applied by the physician.

Previous investigations on this matter have relied on pure numerical methods such as finite differences (O'Brien and Mekkaoui, 1993; Rawnsley et al., 1994) and boundary elements (Chan, 1992) or on approximate and exact analytical solutions (Chato, 1980; Huang et al., 1994). Analytical solutions allow conditions that occur during hyperthermia treatments to be studied in closed form and also provide an improved basis for verification of numerical codes. Accordingly, the main contribution of this work is to employ the classical integral transform technique to establish an analytical solution to the two dimensional transient bioheat transfer equation subjected to convective boundary conditions. Following the solution, the influence of blood perfusion on transient heat transfer in several human tissues subjected to a heat source is discussed, leading to conclusions that may aid the planning of hyperthermia treatment. Moreover, this straightforward methodology (Azevedo, 2004; Presgrave, 2005; Presgrave et al., 2005) provides benchmark results for the numerical investigator interested in developing and validating bioheat transfer software.

2. Mathematical formulation

The following heat transfer problem for a rectangular perfused organic tissue is considered in accordance to Pennes (1948) model:

$$\begin{cases} k_t\left(\dfrac{\partial^2 T}{\partial x^2}+\dfrac{\partial^2 T}{\partial y^2}\right)-w_b c_b (T-T_a)+g=\rho_t c_t \dfrac{\partial T}{\partial t} & 0<x<L,\quad 0<y<l,\quad t>0 \\ \dfrac{\partial T}{\partial x}=0 & x=0,\quad t>0 \\ \dfrac{\partial T}{\partial x}=0 & x=L,\quad t>0 \\ -k\dfrac{\partial T}{\partial y}+hT=hT_\infty & y=0,\quad t>0 \\ T=T_p & y=l,\quad t>0 \\ T=T_0 & 0\le x\le L,\quad 0\le y\le l,\quad t=0 \end{cases} \tag{1}$$

The second term on the left hand side of the bioheat transfer equation is a sink term due to the convective effect of capilar vascularization in living tissues while the third one is a source term representing a combined effect of both the internal metabolic heat generation and the external irradiation. This mathematical model aims to predict the temperature levels in a perfused tissue subjected to a hyperthermic treatment based on an external heating device.

The boundary conditions are taken as prescribed constant temperature at the top and convective heat transfer with an external medium at the bottom. This latter boundary condition attempts to simulate the heat transfer between the tissue and an adjoint large blood vessel. Initial temperature is considered constant.

By introducing the following dimensionless variables:

$$\theta=\frac{T-T_p}{g_0\frac{L^2}{k_t}},\quad G=\frac{g}{g_0},\quad A=\frac{l}{L},\quad X=\frac{x}{L},\quad Y=\frac{y}{L},\quad \tau=\frac{\alpha_t t}{L^2},\quad Pf=\frac{w_b c_b L^2}{k_t},\quad Bi=\frac{hL}{k_t}\ . \tag{2}$$

Problem (1) is given in dimensionless form as:

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}-Pf\,\theta+G=\frac{\partial\theta}{\partial\tau} \qquad 0<X<1,\quad 0<Y<A,\quad \tau>0 \tag{3.a}$$

$$\frac{\partial\theta}{\partial X}=0 \qquad X=0,\quad \tau>0 \tag{3.b}$$

$$\frac{\partial\theta}{\partial X}=0 \qquad X=1,\quad \tau>0 \tag{3.c}$$

$$-\frac{\partial\theta}{\partial Y}+Bi\theta=Bi\theta_\infty \qquad Y=0,\quad \tau>0 \tag{3.d}$$

$$\theta=0 \qquad Y=A,\quad \tau>0 \tag{3.e}$$

$$\theta=T_0 \qquad 0\le X\le 1,\quad 0\le Y\le A,\quad \tau=0 \tag{3.f}$$

This is a particular case of the so-called class I problem (Mikhailov and Özisik, 1984), and may be solved by the classical Integral Transform Technique (Özisik, 1980) as described in the next section. Dimensionless time τ is also referred as Fourier number (Fo) in the literature. However, for the sake of conciseness, the symbol τ shall be employed here.

3. Solution

The temperature field is expressed in the form of an expansion in terms of eigenfunctions :

$$\theta(X,Y,\tau)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{ij}(\tau)\,\psi_i(X)\,\lambda_j(Y) \tag{4}$$

where eigenfunctions $\psi_i(X)$ and $\lambda_j(Y)$ are solutions to eigenvalues problems in the X and Y directions, respectively, obtained as follows:

First, in order to extract the desired Sturm-Liouville problems basic to eq. (4), the homogeneous version of problem (3) is considered:

$$\frac{\partial^2\theta}{\partial X^2}+\frac{\partial^2\theta}{\partial Y^2}-Pf\theta+=\frac{\partial\theta}{\partial\tau} \qquad 0<X<1, \quad 0<Y<A, \quad \tau>0 \tag{5.a}$$

$$\frac{\partial\theta}{\partial X}=0 \qquad X=0, \quad \tau>0 \tag{5.b}$$

$$\frac{\partial\theta}{\partial X}=0 \qquad X=1, \quad \tau>0 \tag{5.c}$$

$$-\frac{\partial\theta}{\partial Y}+Bi\theta=0 \qquad Y=0, \quad \tau>0 \tag{5.d}$$

$$\theta=0 \qquad Y=A, \quad \tau>0 \tag{5.e}$$

$$\theta=T_0 \qquad 0\le X\le 1, \quad 0\le Y\le A, \quad \tau=0 \tag{5.f}$$

Then, by applying the method of separation of variables in problem (5), two eigenvalues problems are obtained. The first one is an eigenvalue problem in the X direction,

$$\frac{d^2\psi_i}{dX^2}+\mu_i^2\psi_i(X)=0 \qquad 0<X<1 \tag{6.a}$$

$$\left.\frac{d\psi_i}{dX}\right|_{X=0}=0 \tag{6.b}$$

$$\left.\frac{d\psi_i}{dX}\right|_{X=1}=0 \tag{6.c}$$

the eigenfunction of the above problem is found to be:

$$\psi_i(X)=\cos\mu_i X \tag{7}$$

where,

$$\mu_i=(i-1)\pi \qquad i=1,2,3,4,...$$

and the norm is

$$N_i=\int_0^1 \psi_i^2(X)dX=2 \tag{8}$$

Moreover, due to the boundary condition of the second type at both x extremes, $\mu_0=0$ is also a solution to problem (6) and the corresponding eigenfunction and norm are:

$$\psi_0(X)=1 \tag{9}$$

and

$$N_0=1 \tag{10}$$

The second eigenvalue problem is:

$$\frac{d^2\lambda_j}{dY^2} + (\gamma_j^2 - Pf)\lambda_j(Y) = 0 \qquad 0 < Y < A \tag{11a}$$

$$-\frac{d\lambda}{dY} + Bi\lambda_j(Y) = 0 \qquad Y = 0 \tag{11b}$$

$$\lambda_j(Y) = 0 \qquad Y = A\ . \tag{11c}$$

By considering

$$\gamma_j^2 - Pf = \beta_j^2 \tag{12}$$

the solution to problem (11) above is:

$\lambda_j(Y) = \text{sen}\,\beta_j(A - Y)$, where β_j are the positive roots of the following transcendental equation

$$\beta_j \cot\beta_j A = -Bi \tag{13}$$

and the norm M_j is given by $\int_0^A \lambda_j^2(Y)dY$, i.e.,

$$M_j = \frac{A(B_j^2 + Bi^2) + Bi}{2(B_j^2 + Bi^2)} \qquad j = 1,2,3,4... \tag{14}$$

Transcendental equation (13) may be solved by the Bissection Method or by well-established mathematical routines such as DZEBREN/IMSL (1999).

Next, the eigenvalues problems obtained above are used to yield the transformations in the X and Y directions and its respectives inversions.

Thus, eq. (4) is operated on by $\int_0^1 \psi_m(X)\ dX$:

$$\int_0^1 \theta(X,Y,\tau)\ \psi_m(X)dX = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{ij}(\tau)\lambda_j(Y)\int_0^1 \psi_m(X)\psi_i(X)\ dX \tag{15}$$

Recalling the orthogonality property of the eigenfunctions $\psi_m(X)$, which is written as:

$$\int_0^{As} \psi_i(X)\psi_m(X)\ dX = \begin{cases} 0, & \text{if} \quad i \neq m \\ M_j, & \text{if} \quad i = m \end{cases} \tag{16}$$

Equation (15) results in,

$$\int_0^1 \theta(X,Y,\tau)\ \psi_m(X)dX = \sum_{j=1}^{\infty} C_{mj}(\tau)\lambda_j(Y)N_m \tag{17}$$

Now, eq. (17) is operated on with $\int_0^A \lambda_n(Y)\ dY$:

$$\int_0^A\int_0^1 \theta(X,Y,\tau)\ \psi_m(X)\lambda_n(Y)dXdY = \sum_{j=1}^{\infty} C_{mj}(\tau)N_m\int_0^A \lambda_j(Y)\lambda_n(Y)dY \tag{18}$$

Again, by the orthogonality property of the eigenfunctions $\lambda_n(Y)$:

$$\int_0^A \int_0^1 \theta(X,Y,\tau)\ \psi_m(X)\lambda_n(Y)dXdY = C_{mn}(\tau)N_m M_n \tag{19}$$

and therefore $C_{mn}(\tau)$ is obtained as

$$C_{mn}(\tau) = \frac{1}{N_m M_n} \int_0^A \int_0^1 \theta(X,Y,\tau)\ \psi_m(X)\lambda_n(Y)dXdY \tag{20}$$

Upon substitution of $C_{mn}(\tau)$ into Eq. (4), it is obtained:

$$\theta(X,Y,\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\psi_i(X)}{N_i} \frac{\lambda_j(Y)}{M_j} \int_0^A \int_0^1 \theta(X,Y,\tau)\ \psi_i(X)\lambda_j(Y)dXdY \tag{21}$$

Now, let

$$\overline{\theta_i}(Y,\tau) = \int_0^1 \psi_i(X)\ \theta(X,Y,\tau)\ dX \tag{22}$$

This integral represents the transform of temperature distribution $\theta(X,Y,\tau)$ in the X direction, resulting into a transformed potential $\overline{\theta_i}(Y,\tau)$.

The substitution of eq. (22) into eq. (21) yields:

$$\theta(X,Y,\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\psi_i(X)}{N_i} \frac{\lambda_j(Y)}{M_j} \int_0^A \overline{\theta_i}(Y,\tau)\ \lambda_j(Y)dY \tag{23}$$

The above equation is the inversion for the transform, which takes the transformed temperature $\overline{\theta_i}(Y,\tau)$ back to its original form $\theta(X,Y,\tau)$. Now, let :

$$\overline{\overline{\theta_{ij}}}(\tau) = \int_0^A \lambda_j(Y)\ \overline{\theta_i}(Y,\tau)dy \tag{24}$$

This integral is the transform of $\overline{\theta_i}(Y,\tau)$ in the Y direction, resulting in a second transform $\overline{\overline{\theta}}_{ij}(\tau)$, now a function of dimensionless time only.

Upon substitution of eq. (24) into eq. (23), a recovery formulae for the original temperature $\theta(X,Y,\tau)$ is obtained, which performs the inversion of the transformed potentials into the X and Y directions, $\overline{\overline{\theta}}_{ij}(\tau)$:

$$\theta(X,Y,\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\psi_i(X)}{N_i} \frac{\lambda_j(Y)}{M_j} \overline{\overline{\theta_{ij}}}(\tau) \tag{25}$$

Having established the desired integral-transform pair for problem (3), the next step is to rewrite the original formulation in terms of $\overline{\overline{\theta}}_{ij}(\tau)$. Thus, eq. (3a) is multiplied by $\psi_i(X)$ and integration from X = 0 to X = 1 is performed, giving:

$$\int_0^1 \psi_i(X)\frac{\partial^2\theta}{\partial X^2}dX + \int_0^1 \psi_i(X)\frac{\partial^2\theta}{\partial Y^2}dX - \int_0^1 \psi_i(X)\ Pf\ \theta\ dX + \int_0^1 \psi_i(X)\ G\ dX = \int_0^1 \psi_i(X)\frac{\partial\theta}{\partial\tau}dX \tag{26}$$

By making use of eq. (22), this expression is rewritten as:

$$\int_0^1 \psi_i(X)\frac{\partial^2\theta}{\partial X^2}dX + \frac{\partial^2\overline{\theta_i}}{\partial Y^2} - Pf\,\overline{\theta_i} + \overline{G_i} = \frac{\partial\overline{\theta_i}}{\partial\tau} \tag{27}$$

where it is defined,

$$\overline{G_i} = \int_0^1 \psi_i(X)\ G\ dX \tag{28}$$

By solving the integral in eq. (27) by parts, one obtains:

$$\left[\psi_i(X)\frac{\partial\theta}{\partial X}-\theta\,\frac{d\psi_i}{dX}\right]_0^1+\int_0^1\theta\frac{d^2\psi_i(X)}{dX^2}dX+\frac{\partial^2\overline{\theta_i}}{\partial Y^2}-Pf\,\overline{\theta_i}+\overline{G_i}=\frac{\partial\overline{\theta_i}}{\partial\tau} \tag{29}$$

The first term of this equation is evaluated by multiplying boundary condition eq. (5b) by $\psi_i(0)$:

$$\psi_i(0)\left.\frac{\partial\theta}{\partial X}\right|_{X=0}=0 \tag{30}$$

and boundary condition eq. (6b) by $\theta(0,Y,\tau)$:

$$\left.\frac{d\psi_i(X)}{dX}\right|_{X=0}\theta(0,Y,\tau)=0 \tag{31}$$

and subtracting eq. (30) from eq. (31) to give:

$$\psi_i(0)\left.\frac{\partial\theta}{\partial X}\right|_{X=0}-\left.\frac{d\psi_i(X)}{dX}\right|_{X=0}\theta(0,Y,\tau)=0 \tag{32}$$

Analogously, by using boundary conditions at X = 1 from eq. (5) and eq. (6), the following relation is established:

$$\psi_i(1)\left.\frac{\partial\theta}{\partial X}\right|_{X=1}-\left.\frac{d\psi_i(X)}{dX}\right|_{X=1}\theta(1,Y,\tau)=0 \tag{33}$$

Next, eq. (32) and eq. (33) are substituted into eq. (29) resulting in:

$$0+\int_0^1\theta\frac{d^2\psi_i(X)}{dX^2}dX+\frac{\partial^2\overline{\theta_i}}{\partial Y^2}-Pf\,\overline{\theta_i}+\overline{G_i}=\frac{\partial\overline{\theta_i}}{\partial\tau} \tag{34}$$

Now, the integral in eq. (34) above is evaluated by operating on eq. (6a) with $\int_0^1\theta_i dX$ and making use of transform eq. (22):

$$\int_0^1\theta_i\frac{d^2\psi_i(X)}{dX^2}dX=-\mu_i^2\int_0^1\theta_i\;\psi_i(X)dX=-\mu_i^2\overline{\theta_i} \tag{35}$$

Next, eq. (35) is substituted into eq. (34) resulting in:

$$-\mu_i^2\overline{\theta_i}+\frac{\partial^2\overline{\theta_i}}{\partial Y^2}-Pf\,\overline{\theta_i}+\overline{G_i}=\frac{\partial\overline{\theta_i}}{\partial\tau} \tag{36}$$

Equation (36) is a partial differential equation in Y and τ . In order to transform the problem in the Y direction, eq. (36) is multiplied by $\lambda_j(Y)$ and integration from Y = 0 to Y = A is performed, resulting in:

$$-\mu_i^2\int_0^A\lambda_j(Y)\overline{\theta_i}dY+\int_0^A\lambda_j(Y)\frac{\partial^2\overline{\theta_i}}{\partial Y^2}dY-Pf\int_0^A\lambda_j(Y)\overline{\theta_i}dY+\int_0^A\lambda_j(Y)\,\overline{G_i}dY=\int_0^A\lambda_j(Y)\frac{\partial\overline{\theta_i}}{\partial\tau}dY \tag{37}$$

By utilizing transform eq. (24) and by defining $\overline{\overline{G_{ij}}}=\int_0^A\lambda_j(Y)\,\overline{G_i}dY$, it is obtained:

$$-\mu_i^2\overline{\overline{\theta}}_{ij}(\tau)+\int_0^A\lambda_j(Y)\frac{\partial^2\overline{\theta}_i}{\partial Y^2}dY-Pf\,\overline{\overline{\theta}}_{ij}(\tau)+\overline{\overline{G}}_{ij}=\frac{d\overline{\overline{\theta}}_{ij}(\tau)}{d\tau} \tag{38}$$

Equation (38) is integrated by parts to yield :

$$-\mu_i^2\overline{\overline{\theta}}_{ij}(\tau)+\left[\lambda_j(Y)\frac{\partial\overline{\theta}_i}{\partial Y}-\overline{\theta}_i\frac{d\lambda_j}{dY}\right]_0^A+\int_0^A\overline{\theta}_i\frac{d^2\lambda_j}{dY^2}dY-Pf\,\overline{\overline{\theta}}_{ij}(\tau)+\overline{\overline{G}}_{ij}=\frac{d\overline{\overline{\theta}}_{ij}(\tau)}{d\tau} \tag{39}$$

The third term of this equation is evaluated with the aid of the eigenvalue problem in the Y direction. Equation (11) is operated on with $\int_0^A\overline{\theta}_i\ dY$ and the transform eq. (24) is used to give:

$$\int_0^A\overline{\theta}_i\frac{d^2\lambda_j}{dY^2}dY=-\beta_j^2\int_0^A\overline{\theta}_i\lambda_j(Y)dY=-\beta_j^2\overline{\overline{\theta}}_{ij}(\tau) \tag{40}$$

After substituting eq. (40) above into eq. (39), it is found that:

$$-\mu_i^2\overline{\overline{\theta}}_{ij}(\tau)+\left[\lambda_j(Y)\frac{\partial\overline{\theta}_i}{\partial Y}-\overline{\theta}_i\frac{d\lambda_j}{dY}\right]_0^A-\beta_j^2\overline{\overline{\theta}}_{ij}(\tau)-Pf\,\overline{\overline{\theta}}_{ij}(\tau)+\overline{\overline{G}}_{ij}=\frac{d\overline{\overline{\theta}}_{ij}(\tau)}{d\tau} \tag{41}$$

The second term of eq. (41) above is evaluated by multiplying the boundary condition at Y = 0 for problem (11) by $\overline{\theta}_i(0,\tau)$:

$$-\overline{\theta}_i(0,\tau)\left.\frac{d\lambda_j}{dY}\right|_{Y=0}+Bi\,\overline{\theta}_i(0,\tau)\,\lambda_j(0)=0 \tag{42}$$

and the boundary condition at Y = 0 for problem (3) by $\lambda_j(0)$:

$$-\lambda_j(0)\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0}+Bi\,\theta\,\lambda_j(0)=Bi\theta_\infty\lambda_j(0) \tag{43}$$

Next, the X direction transform is applied to give:

$$\lambda_j(0)\left.\frac{\partial\overline{\theta}_i}{\partial Y}\right|_{Y=0}-Bi\,\lambda_j(0)\overline{\theta}_i=-Bi\,\theta_\infty\lambda_j(0)\int_0^1\psi_i dX \tag{44}$$

By substituting, now, eq. (42) into eq. (44), the following expression at Y = 0 is obtained:

$$\lambda_j(0)\left.\frac{\partial\overline{\theta}_i}{\partial Y}\right|_{Y=0}-\overline{\theta}_i(0,\tau)\left.\frac{d\lambda_j}{dY}\right|_{Y=0}=-Bi\,\theta_\infty\lambda_j(0)\int_0^1\psi_i dX \tag{45}$$

By multiplying eq. (11c) by $\left.\frac{\partial\overline{\theta}_i}{\partial Y}\right|_{Y=A}$ and eq. (3e) by $\frac{d\lambda_j(A)}{dY}$ results in, respectively:

$$\lambda_j(A)\left.\frac{\partial\overline{\theta}_i}{\partial Y}\right|_{Y=A}=0 \tag{46}$$

and

$$\theta(X,A,\tau)\frac{d\lambda_j(A)}{dY}=0 \tag{47}$$

The X direction transform is applied in eq. (47) above furnishing:

$$\int_0^1 \psi_i \theta(X,A,\tau) \frac{d\lambda_j(A)}{dY} dX = 0 \tag{48}$$

or,

$$\left.\frac{d\lambda_j}{dY}\right|_{Y=A} \overline{\theta_i}(A,\tau) = 0 \tag{49}$$

By subtracting eq. (46) from eq. (49), the following expression at Y = A results:

$$\lambda_j(A) \left.\frac{\partial \overline{\theta_i}}{\partial Y}\right|_{Y=A} - \left.\frac{d\lambda_j}{dY}\right|_{Y=A} \overline{\theta_i}(A,\tau) = 0 \tag{50}$$

Finally, substituting eq. (45) and eq. (50) into eq. (41),

$$-\mu_i^2 \overline{\overline{\theta_{ij}}}(\tau) - \beta_j^2 \overline{\overline{\theta_{ij}}}(\tau) + \mathrm{Bi}\theta_\infty \lambda_j(0) \int_0^1 \psi_i(X) dX - \mathrm{Pf}\, \overline{\overline{\theta_{ij}}}(\tau) + \overline{\overline{G_{ij}}} = \frac{d\overline{\overline{\theta_{ij}}}(\tau)}{d\tau} \tag{51}$$

and making use of definition eq. (12):

$$-\mu_i^2 \overline{\overline{\theta_{ij}}}(\tau) - \beta_j^2 \overline{\overline{\theta_{ij}}}(\tau) + \mathrm{Bi}\theta_\infty \lambda_j(0) \int_0^1 \psi_i(X) dX - \gamma_j^2 \overline{\overline{\theta_{ij}}}(\tau) + \beta_j^2 \overline{\overline{\theta_{ij}}}(\tau) + \overline{\overline{G_{ij}}} = \frac{d\overline{\overline{\theta_{ij}}}(\tau)}{d\tau} \tag{52}$$

Therefore, a system of ordinary differential equations as a function of τ results, for the determination of the transformed original problem. This system is rewritten in the following form:

$$\frac{d\overline{\overline{\theta}}_{ij}(\tau)}{d\tau} + (\mu_i^2 + \gamma_j^2)\, \overline{\overline{\theta_{ij}}}(\tau) = \overline{\overline{P_{ij}}} \tag{53}$$

where,

$$\overline{\overline{P_{ij}}} = \mathrm{Bi}\ \theta_\infty\ \lambda_j(0) \int_0^1 \psi_i(X) dX + \int_0^A \int_0^1 \lambda_j(y) \psi_i(X)\, G\, dX\, dY \tag{54}$$

The transformed initial condition is given by :

$$\overline{\overline{\theta_{ij}}}(0) = \int_0^A \int_0^1 \lambda_j(y)\, \psi_i(X)\, \theta(X,Y,0)\, dX\, dY \tag{55}$$

The exact solution for this decoupled system of ordinary differential equations is:

$$\overline{\overline{\theta}}_{ij}(\tau) = \frac{\overline{\overline{P_{ij}}}}{\mu^2{}_i + \gamma_j{}^2} - \frac{e^{-(\mu_i^2+\gamma_j^2)\tau}}{\mu^2{}_i + \gamma_j{}^2} \overline{\overline{P_{ij}}} + e^{-(\mu_i^2+\gamma_j^2)\tau}\, \overline{\overline{\theta}}_{ij}(0) \tag{56}$$

Now, the original temperature field $\theta(X,Y,\tau)$ is recast by successive application of inversion in the X and Y directions, eq.(29) and eq. (27), resulting in:

$$\theta(X,Y,\tau) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\psi_i(X)}{N_i} \frac{\lambda_j(Y)}{M_j} \left[\frac{\overline{\overline{P_{ij}}}}{\mu^2{}_i + \gamma_j{}^2} - \frac{e^{-(\mu_i^2+\gamma_j^2)\tau}}{\mu^2{}_i + \gamma_j{}^2} \overline{\overline{P_{ij}}} + e^{-(\mu_i^2+\gamma_j^2)\tau}\, \overline{\overline{\theta}}_{ij}(0) \right] \tag{57}$$

4. Results and discussion

In this section, numerical simulations are presented in order to analyze the effects of the perfusion term on Pennes equation for several tissues subjected to external heating and metabolic generation. According to the dimensionless variables employed here, the initial (θ_0) and ambient (θ_∞) temperatures are functions of both tissue and heat generation. In every case studied, the characteristic dimension L is 0.03 m and T_a = 36.5 °C. The adopted value for the external heat source is 50,000 W/m^3 and metabolic heat generation rate is 33,800 W/m^3, (Deng and Liu, 2002). As a result, g = 83,800 W/m^3 in all computations performed in this section. Perfusion is regarded as constant and its value varies with the tissue being considered. Thermophysical properties of blood and tissues are also considered constant. Accordingly, blood density and specific heat are taken as ρ_b = 1060 kg/m^3 and c_b = 3720 J/kgK, respectively, (Brix et al., 2002), while perfusion and thermal conductivity for several tissues are listed in Tab. (1).

Table 1. Thermophysical properties for several tissues (subscript t)

Reference	Blood perfusion w_b $[m_b{}^3 s^{-1} m_t{}^{-3}]$	Density ρ_t $[kgm^{-3}]$	Specific heat c_t $[Jkg^{-1}k^{-1}]$	Thermal conductivity k_t $[Wm^{-1}k^{-1}]$	Pf Eq.(2)
Chan (1992) - ficticious tissue	0.00001	1000	4185	0.50	0.1
Brix et al. (2002) – adipose tissue	0.00050	950	3100	0.27	5
Jiang et al.(2002) – inner tissue	0.00125	1000	4000	0.50	10
Jiang et al.(2002) – subcutaneous tissue	0.00125	1000	2500	0.19	15
Brix et al. (2002) – liver	0.01500	1060	3600	0.52	100
Brix et al. (2002) – kidney	0.06100	1050	3700	0.54	400

For the sake of computations, dimensionless perfusion (Pf) values computed with above data are rounded to the figures indicated in the table.

Dimensionless temperature distributions are obtained from Eq. (57) upon truncation of the infinite sums to a sufficiently large order that ensures a converged result of at least three significant digits. Convergence is found to be fast and typically no more than 30 terms are needed in the summations. A complete discussion regarding the convergence behavior of the infinite series is found in Azevedo (2004).

Figures (1) - (3) show dimensionless temperature distributions at the center of the tissue as a function of dimensionless time and dimensionless perfusion for the situation of Bi = 5, G = 1, θ_0 = 0.003 and θ_∞ = 0.001, and three different aspect ratios, namely, 0.25, 0.50 and 1.

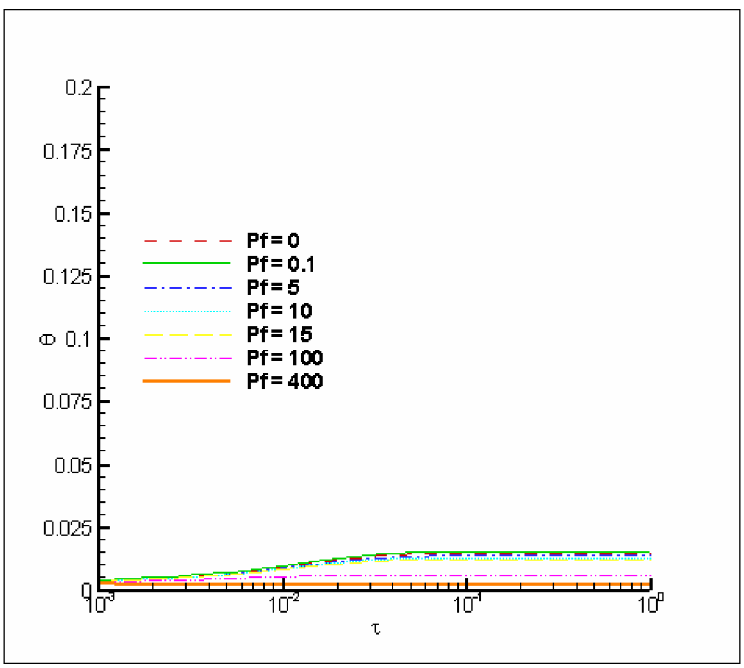

Figure 1. Dimensionless temperature distribution: Bi = 5, G = 1, θ_0 = 0.003 , θ_∞ = 0.001, A = 0.25.

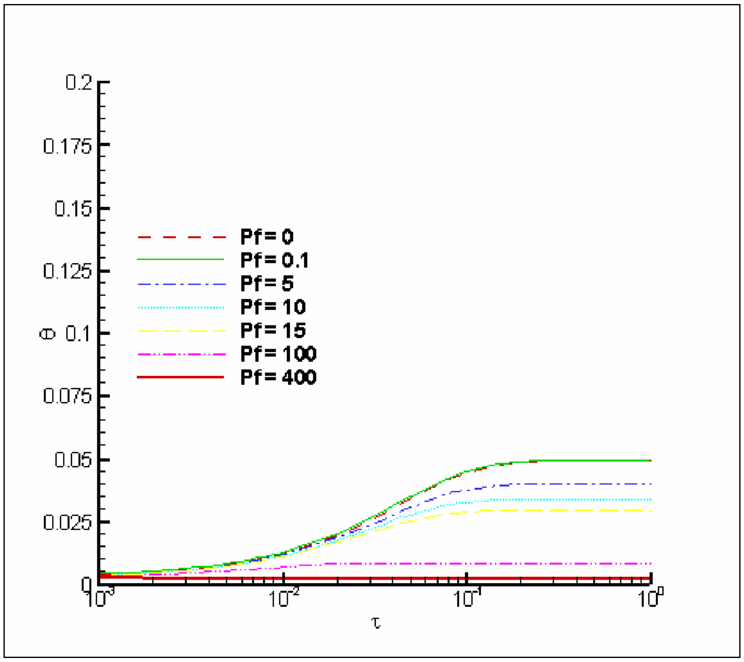

Figure 2. Dimensionless temperature distribution: Bi = 5, G = 1, θ_0 = 0.003 , θ_∞ =0.001, A = 0.50.

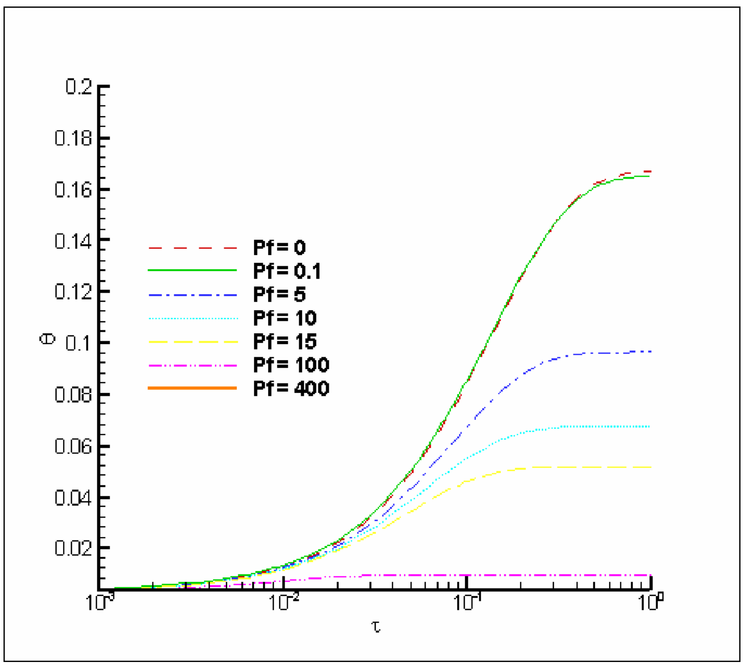

Figure 3. Dimensionless temperature distribution: Bi = 5, G = 1, θ_0 = 0.003 , θ_∞ =0.001, A = 1.0.

An inspection of the above graphs reveal certain interesting trends regarding the transient temperature distributions in an applied situation such as a hyperthermia treatment of a cancerous tissue. As already mentioned earlier, in such situations, it is desired to achieve a certain degree of temperature, usually around 44 °C, in order to destroy the malignant cells. Figure (1) shows that the temperature distributions are only slightly above its initial condition for the case of a low aspect ratio tissue such as A = 0.25 for all the simulated perfusion coefficients suggesting that the desired threshold for hyperthermia is not being achieved. On the other hand, for aspect ratios greater than 0.25, temperature levels are significantly detached from their initial conditions. In fact, Fig. (2) and Fig. (3) illustrate this tendency when dimensionless times greater than 0.01 are considered. Also, the role of the perfusion coefficient in the temperature distribution can be inferred from these figures. Consistent with its role as a sink term, a high perfusion rate results in a severe temperature decrease. This behavior can be better observed in Fig. (3) by analyzing the curves related to $P_f = 0.0$ and 400, which simulate a situation without any perfusion effects and another one in which an extreme heat sink due to blood flow is considered. For the steady-state situation, despite the action of the external heat source, the $P_f = 400$ curve is slightly above the initial condition whereas the $P_f = 0$ situation achieves a dimensionless temperature of around 0.16. Also worth mentioning is the fact that for dimensionless times less than 0.01, the temperature in the center of the targeted area does not seem to be strongly influenced by the perfusion coefficient. If high temperatures are desired, as in the case of hyperthemia, certainly the exposure time should be much greater than 0.01.

Next, we present some results in dimensional form in order to obtain a better physical perception of the role of the various above mentioned quantities, in the temperature fields related to the bioheat transfer problem in question. Table (2) shows the dimensional initial temperature together with its steady-state value in the geometrical center of the targeted area for three aspect ratios and for several tissues as described in Tab. (1). Despite the variation of the thermophysical properties, an analysis of Tab. (2) suggests that it is a good assumption to consider the same initial condition and external temperature for all the cases reported. On the other hand, a fixed Biot number does not imply in the same convective heat transfer since here, the thermal conductivity of a certain tissue may be quite different from another one. For a fixed perfusion coefficient, temperature levels monotonically increases for increasingly aspect ratios. A closer observation shows that for liver and kidney, steady state temperature distributions remain very close to their initial levels for the external heat source of 50,000 W/m^3. This observation suggest that the hypertermia treatment for these tissues is inefficient for the value of external heating adopted here as the results obtained in the simulations indicate that the temperature is below 44 °C. Also, target areas with an aspect ratio of 0.25 show steady state temperature below the 44 °C threshold. As a result, with such perfusion coefficients, hyperthemia will only be an effective treatment for situations which involve tissues with aspects ratios greater or equal to 0.5. In addition, an inspection of this table shows the role of the combined effects of the perfusion, conduction and convection in the bioheat transfer process. For instance, although the perfusion coefficient for the adipose tissue is 50 times greater than the one reported by Chan (1992), its steady state temperature is actually about 2 °C higher for the A=0.25 situation. This apparent contradiction is better understood by noticing that the thermal conductivity of the adipose tissue and the convective coefficient are considerably smaller than those reported by Chan (1992) implying that the convection and conduction effects for this situation are not dominant, resulting in a low heat transfer rate to the external environment. Therefore, the energy balance indicates that the source term combined with the low convection and poor conduction dominates over the perfusion heat sink. On the other hand, a comparison between the results reported by Chan (1992) and our simulations regarding the inner tissue show that for this case the temperature levels decrease as expected since the thermal conductivity, initial and environmental temperatures and convective coefficients are the same in both cases. Since the only difference is in the perfusion coefficient, it is naturally expected that the inner tissue should present a smaller steady-state temperature.

Table2 - Initial and steady-state temperatures for Bi = 5.0; G = 1.0; $\theta_0 = 0.003$; $\theta_\infty = 0.001$.

Reference	T_0 [^{0}C]	T_∞ [^{0}C]	h [W m^{-2} K^{-1}]	$T_{\frac{1}{2}f}$ [^{0}C] $A = 0.25$	$T_{\frac{1}{2}f}$ [^{0}C] $A = 0.50$	$T_{\frac{1}{2}f}$ [^{0}C] $A = 1.00$
Chan (1992) - ficticious tissue	36.95	36.65	83.3	38.76	43.92	61.32
Brix et al. (2002) – adipose tissue	37.34	36.78	45.0	40.43	47.81	63.51
Jiang et al. (2002) – inner tissue	36.95	36.65	83.3	38.49	41.67	46.71
Jiang et al. (2002) – subcutaneous tissue	37.69	36.90	31.7	41.43	48.29	57.08
Brix et al. (2002) – liver	36.94	36.65	86.7	37.41	37.79	37.94
Brix et al. (2002) – kidney	36.92	36.64	90.0	36.82	36.85	36.85

Finally, Table (3) shows the equivalent dimensional times for $\tau = 0.1$ together with the respective central point temperatures for the high aspect ratio situation reported in this contribution. Once again, an inspection of these values indicate that the hyperthermia threshold is not achieved for the liver and kidney tissues. However, for the other biological tissues considered in our analysis, temperature levels are indeed above 44 °C in a time frame greater than 10 and less than 20 minutes for the external heat source mentioned earlier. Such results are believed to be relevant for the planning of a successful treatment of malignant tumors by hyperthermia.

Table 3 . Equivalent dimensional times for τ = 0.1 and respective temperatures at the center of the tissue. Bi = 5.0; G = 1.0; θ_0 = 0.003; θ_∞ =0.001; A = 1.0

Reference	Time	$T_{\frac{1}{2}f}\ [^0C]$
Chan (1992) - ficticious tissue	10min 54sec	49.24
Brix et al. (2002) – adipose tissue	16min 21sec	55.48
Jiang et al. (2002) – inner tissue	12min	44.88
Jiang et al. (2002) – subcutaneous tissue	19min 44sec	54.86
Brix et al. (2002) – liver	11min	37.94
Brix et al. (2002) – kidney	10min 47sec	36.85

5. Conclusions

In conclusion, a straightforward methodology based on the classical integral transform technique is devised to aid physicians throughout the decision making process regarding the use of high energy sources for the destruction of cancerous cells located in various parts of the human body. Although the methodology here reported was only tested in the cartesian system, our research points out that more elaborate geometries can also be successful tackled by the same approach with equal mathematical simplicity. For example, the same solution procedure was successfully applied to the temperature distribution in a human limb subjected to a skin burn and also to the selective cooling of the human brain which is a medical procedure designed to aid patients in the immediate moments following an ischemic trauma (Presgrave, 2005).

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